1) Structure function power spectra

In my lecture 15 xcorr_fft notes I derived the relationship between the covariance and the power spectrum. Below I'll start with those equations, but move from signals that vary in time to signals that vary in space (satellite images) and from one dimension to two dimensions.

Start with the 1-d Fourier Transform of the vertical velocity:

$$W(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w e^{ikx} dx \tag{1}$$

(where I've switched to spatial wavenumber=$k = 2\pi/\lambda$ instead of frequency f for this example – see Appendix B of Cushman-Roisin for a good discussion of wavenumber and 2-d plane waves.)

Following the lecture 15 notes, if we define the one-dimensional spatial autocovariance as:

$$cov_{ww}(r) = \Gamma(r) = \int_{-\infty}^{\infty} w(x)w(x+r)dr \tag{2}$$

(note the stationarity assumption – there is no dependence on our starting point x. Also note that I've already removed the mean from w).

As in equation lecture 15.2, take the Fourier transform of (2) and rearrange the limits of integration to get power spectrum:

$$E(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikr}\Gamma(r)dr \tag{3}$$

$$= W(k)W^*(k) = |W|^2 \tag{4}$$

where the * denotes the complex conjugate. If the integral

$$\int_{0}^{\infty} E(k)dk < \infty, \tag{5}$$

exists then the field is homogeneous (i.e. spatially stationary) and

$$\Gamma(0) = \sigma_w^2 = \int_{-\infty}^{\infty} E(k)dk \tag{6}$$

as in equation 15.10.

Also note that the inverse transform to get the autocorrelation from the power spectrum is:

$$\Gamma(r) = \int_{-\infty}^{\infty} e^{-ikr}E(k)dk \tag{7}$$

This relationship between the correlation and the power spectrum is called the *Wiener-Khinchin* theorem (Numerical Recipes 12.0.12)

1.a) 1-d structure function

How does this relate to the structure function? The definition of the one dimensional 2nd order structure function is:

$$S_2(x,r) = \langle (f(x+r) - f(x))^2 \rangle \tag{8}$$

where the angled brackets denote an ensemble average. If we write $f(x) = f_1$ and $f(x+r) = f_2$, then (8) is equivalent to:

$$S_2 = \langle (f_2 - f_1)^2 \rangle = \langle f_2^2 \rangle + \langle f_1^2 \rangle - 2\langle f_2 f_1 \rangle \tag{9}$$

If the fields are stationary, then $\langle f_1^2 \rangle = \langle f_2^2 \rangle =$ (the variance of $f(x)$) $= \langle f^2 \rangle = \Gamma(0)$ and in the same way we can write $\langle f_2 f_1 \rangle = \langle f(x+r)f(x) \rangle = \Gamma(r)$. Therefore (9) becomes:

$$S_2 = 2\,(\Gamma(0) - \Gamma(r)) \tag{10}$$

Using (7) and (6) we can rewrite (10) as

$$\begin{aligned} S_2 = 2\int_{-\infty}^{\infty} \left(1 - e^{-ikr}\right) E(k)dk &= 2\int_{-\infty}^{\infty} (1 - (\cos(-kr) + i\sin(-kr)))\, E(k)dk \\ &= 2\int_{-\infty}^{\infty} (1 - \cos(-kr))\, E(k)dk \end{aligned} \tag{11}$$

where we have dropped the sin term since $E(k)$ is even and $\sin(-kr)$ is odd, so that their product will integrate to 0 from $-\infty$ to ∞.

1.b) Extension to 2 dimensions

The main change in moving to two dimensions is that position and wavenumber are both vectors. In this case the structure function $S_p(\vec{\boldsymbol{x}}, \vec{\boldsymbol{r}})$ of order p of a field $\tau(\vec{\boldsymbol{x}})$ is given by:

$$S_p(\vec{\boldsymbol{x}}, \vec{\boldsymbol{r}}) = \langle |\tau(\vec{\boldsymbol{x}} + \vec{\boldsymbol{r}}) - \tau(\vec{\boldsymbol{x}})|^p \rangle, \tag{12}$$

where $\mathbf{x}$ is the position, $\mathbf{r}$ is a lag with respect to $\mathbf{x}$, and the angled braces, $\langle \cdot \rangle$, indicate an ensemble average. The second-order structure function S_2 is obtained by setting $p = 2$ in (12). The spectral density, $\Psi(\mathbf{k})$, of a statistically homogeneous (but possibly anisotropic) two-dimensional random field, $\tau(\vec{\boldsymbol{x}})$, is defined by

$$\Psi(\vec{\boldsymbol{k}}) = (2\pi)^{-2} \int \langle \tau(0)\tau(\mathbf{r}) \rangle\, e^{i\mathbf{k}\cdot\mathbf{r}}\ d^2 r \tag{13a}$$

$$= \left| \int \tau(\vec{\boldsymbol{x}}) e^{i\mathbf{k}\cdot\mathbf{x}}\ d^2 x \right|^2, \tag{13b}$$

where $\mathbf{x} = [x, y]$ is the position, $\langle \tau(0)\tau(\vec{\boldsymbol{r}}) \rangle$ is the (auto) covariance function at lag $\vec{\boldsymbol{r}} = \vec{\boldsymbol{x}}' - \vec{\boldsymbol{x}} = [r_x, r_y]$, the angled braces, $\langle \cdot \rangle$, indicate an ensemble average (i.e., an average over all possible realizations of the field), $\vec{\boldsymbol{k}} = [k_x, k_y]$ is the wave number, and the integrals are over all of the respective space, with $d^2 r = dx\,dy$ in Cartesian coordinates. Eq. (13) is the 2-d version of Wiener-Khinchin formula (equation (4)). which says that the spectral density $\Psi(\vec{\boldsymbol{k}})$ is the modulus squared of the Fourier transform of the field $\tau(\vec{\boldsymbol{x}})$ itself.

So how do we integrate equations like (13)? It helps to switch polar coordinates. Thus

$$\vec{\boldsymbol{k}} = k_x \vec{\boldsymbol{i}} + k_y \vec{\boldsymbol{j}} \tag{14a}$$

becomes

$$k^2 = k_x^2 + k_y^2 \tag{14b}$$

$$k_x = k\cos\theta \tag{14c}$$

$$k_y = k\sin\theta \tag{14d}$$

(check Appendix B of C-R) Similarly for the lag:

$$\vec{\boldsymbol{r}} = x\vec{\boldsymbol{i}} + x\vec{\boldsymbol{j}} \tag{15a}$$

$$r^2 = x^2 + y^2 \tag{15b}$$

$$x = r\cos\phi \tag{15c}$$

$$y = r\sin\phi \tag{15d}$$

With this change we can write:

$$\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} = rk\cos(\theta - \phi) \tag{16a}$$

(vector inner product)

$$dx\,dy = rd\phi dr \tag{16b}$$

(infinitesimal spatial area

$$dk_x\,dk_y = kd\theta dk \tag{16c}$$

(infinitesimal wavenumber area)

$$\tag{16d}$$

We can define a one-dimensional energy spectrum for the two-dimensional field by integrating around a circle. Remembering that $d\vec{\boldsymbol{k}} = kd\theta dk$:

$$E(k) = \int_0^{2\pi} \Psi(\vec{\boldsymbol{k}})\ k\ d\theta, \tag{17}$$

If the field is statistically isotropic, then $\Psi(\vec{\boldsymbol{k}}) = \Psi(k)$, and $E(k) = 2\pi k\Psi(k)$, but even if Ψ depends on θ equation (17) is still valid.

The scalar spectrum $E(k)$, and the second-order structure function $S_2(r)$, of an isotropic two-dimensional field are related by the extended version of the Wiener-Khinchin formula (11):

$$S_2(r) = 2\int \left[1 - \cos(\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}})\right]\ \Psi(\vec{\boldsymbol{k}})\ d\vec{\boldsymbol{k}} \tag{18}$$

We can expand this using (16a)

$$\begin{aligned} S_2(r) &= 2\int [1 - \cos(kr\cos(\theta - \phi))]\ \Psi(\vec{\boldsymbol{k}})\ kd\theta dk \\ &= 2\int_0^{\infty} [1 - J_0(kr)]\ E(k)\ dk, \end{aligned} \tag{19}$$

where we've used Parseval's integral to turn the nested cosine into J_0, which is a Bessel function of the first kind of order zero.

2) Scaling

A field $\tau(x)$ is said to be scaling if the structure function $S_p(r)$ satisfies

$$S_p(\lambda r_1) = \lambda^{\zeta_p} S_p(r_1) \tag{20}$$

over a range of scales, where ζ_p is a scale-invariant exponent, λ is a scaling factor, and r_1 is an arbitrary unit scale. If $S_p(r)$ satisfies (20), then it satisfies the power-law relation

$$S_p(r) \propto r^{\zeta_p}, \tag{21}$$

and it can be described, over the given range of scales, by only two parameters: the exponent ζ_p, and a multiplicative constant.

To be more specific, consider the first order structure function, which is just the equation for the velocity increment δv:

$$\delta v(x, l) = v(x + l) - v(x) \tag{22}$$

Kolmogorov's original 1941 paper started with 4 hypotheses: (source: Turbulence, Urial Frisch, Cambridge University Press 1995, p. 74.)

1. At small scales the velocity increments are isotropic and homogeneous so the statistics of the increments are independent of displacement ρ for ρ smaller than the flow's integral scale l_0 where $l_0 = \int_0^\infty \Gamma(r)dr$.

$$\delta v(x + \rho, l) = \delta v(x, l) \tag{23}$$

2. The velocity increments are scaling (i.e. scale independent), so that

$$\delta v(x, \lambda l) = \lambda^h \delta v(x, l) \tag{24}$$

3. The turbulent flow has a finite non-vanishing (as $Re \to \infty$) mean rate of dissipation per unit mass.

4. In the limit of infinite Reynolds number $Re = UL/\nu$, all small-scale statistical properties are uniquely and universally determined by the scale l and the mean energy dissipation rate ϵ. In other words, the dissipation is independent of the viscosity.

Just as with the $k^{-5/3}$ scaling, we can get $l^{2/3}$ scaling for the second order structure function if it can only depend on l and ϵ. Specifically $(\delta v(x, l))^2$ has units of L^2T^{-2} and ϵ has units of L^2T^{-3}, which leaves only one combination of ϵ and l:

$$\langle(\delta v(x, l))^2\rangle = C\epsilon^{2/3} l^{2/3} \tag{25}$$

This is "Kolmogorov's 2/3 law". Comparing this with (24) shows that we need $h = 1/3$. If that scaling holds for higher order structure functions, then then S_3 should scale as $l^{3/3} = l$. Kolmogorov was able to prove the "four-fifths law"

$$\langle(\delta v(x, l))^3\rangle = -\frac{4}{5}\epsilon l \tag{26}$$

where the $-4/5$ coefficient is exact, and comes directly from the Navier Stokes equation (after a lot of work, see Frisch section 6.2.5).

What about higher order structure functions? Again, just using dimensional analysis gives:

$$S_p(l) = \langle(\delta v(x,l))^p\rangle = C_p \epsilon^{p/3} l^{p/3} \tag{27}$$

If the structure functions obey (27) they are said to have "Kolmogorov 41 scaling". If not then they show "anomalous scaling". In particular, wind tunnel and other measurements show anomalous scaling (see Frisch Figure 8.8).

2.a) Linking the structure function and Fourier transform

Finally, note that if we put $E(k) \propto k^{-5/3}$ into (19) and integrate (neglecting the variation in J_0, we get

$$S_2(r) \approx 2 \int_{k_0}^{\infty} k^{-2/3} dk \sim k_0^{-2/3} \tag{28}$$

where k_0 is the wavenumber of the largest scale. At separation r the largest wavenumber is $k_0 = 2\pi/r$, giving

$$S_2(r) \sim r^{2/3} \tag{29}$$

In agreement with the 2/3 law.

3) Summary

- Measurements of the structure function provide insight into the spatial and temporal variability of turbulence – including a quantitative definition of the intermittency.
- In many cases it is easier to think about variability in the spatial domain, because local eddies have fourier transforms that are spread over many wavenumbers.
- Although I haven't proved it here, all of these structure function results require only stationary increments, not stationary fields. It is much more likely to find stationary increments in the atmosphere.
- The Kolmogorov 2/3 and 4/5 laws, along with the multifractal spectrum, allow different models of turbulence to be tested against data.
- There is still not general agreement of what is determining the intermittancy of inertial range turbulence, but there are several heuristic models that do a decent job of matching the observations.